

Bounds for $GL(3) \times GL(2)$ L -functions and $GL(3)$ L -functions

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Abstract

In this paper, we will give the subconvexity bounds for self dual $GL(3)$ L -functions in the t aspect as well as subconvexity bounds for self dual $GL(3) \times GL(2)$ L -functions in the $GL(2)$ spectral aspect.

1 Introduction

Bounding L -functions on their critical lines is a far-reaching problem in number theory. For a general automorphic L -function, one may apply the Phragmen-Lindeloff interpolation method together with bounds on the L -function in $\Re s > 1$ and $\Re s < 0$ (the latter coming from the functional equation) to give an upper bound for the L -function on the line $\Re s = \frac{1}{2}$. The resulting bound is usually referred to as the convexity bound (or the trivial bound) for the L -function. While the Lindeloff hypothesis is still out of reach, breaking the convexity bounds for L -functions is an interesting problem.

For L -functions of degree one, that is Dirichlet L -functions, such subconvexity estimates are due to Weyl [We] in the t -aspect and Burgess in the q -aspect [Bu]. For degree two L -functions this was achieved in a series of papers by Good [Go], Meurman [Me] and especially Duke, Friedlander and Iwaniec [DFI1, DFI2, DFI3]. Subconvexity for Rankin-Selberg L -functions on $GL(2) \times GL(2)$ were known due to Sarnak [Sa], Kowalski, Michel and Vanderkam [KMV], Michel [Mi], Harcos and Michel [HM], Michel and Venkatesh [MV1], Lau, Liu and Ye [LYL], etc (see the references in [MV2]). Impressive subconvexity estimates for triple L -functions on $GL(2)$ were made by Bernstein and Reznikov [BR], see also Venkatesh [Ve].

Much less is known for subconvexity bounds for L -functions on higher rank groups. In this paper, we establish such subconvexity estimates for Rankin-Selberg L -functions on $GL(2) \times GL(3)$ and L -functions on $GL(3)$. To begin with, let $f(z)$ be a self dual Hecke-Maass form of type (ν, ν) for $SL(3, \mathbb{Z})$, normalized so that the first Fourier coefficient is 1. We define the L -function

$$(1.1) \quad L(s, f) = \sum_{m=1}^{\infty} A(m, 1) m^{-s}.$$

For f and each $u_j(z)$ in an orthonormal basis of even Hecke-Maass forms for $SL(2, \mathbb{Z})$, we define the Rankin-Selberg L -function

$$(1.2) \quad L(s, f \times u_j) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(n, m)}{(m^2 n)^s}.$$

Our main theorem is the following:

Theorem 1.1. *Let f be a fixed self dual Hecke-Maass form for $SL(3, \mathbb{Z})$ and u_j be an orthonormal basis of even Hecke-Maass forms for $SL(2, \mathbb{Z})$ corresponding to the Laplacian eigenvalue $\frac{1}{4} + t_j^2$ with $t_j \geq 0$, then for $\varepsilon > 0, T$ large and $T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}$, we have*

$$(1.3) \quad \sum_j' e^{-\frac{(t_j-T)^2}{M^2}} L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{M^2}} \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt \ll_{\varepsilon, f} T^{1+\varepsilon} M$$

where $'$ means summing over the orthonormal basis of even Hecke-Maass forms.

Remarks 1. The second term in (1.3) comes from the Rankin-Selberg L -function of f and the Eisenstein series on $GL(2)$.

2. By considering the case that f is the minimal Eisenstein series on $GL(3)$, one sees that the sign of the functional equation of $L(s, f \times u_j)$ is $+1$ when u_j is an even Hecke-Maass form and -1 when u_j is an odd Hecke-Maass form for $SL(2, \mathbb{Z})$. For this reason we restrict to even Hecke-Maass forms in (1.3). This feature doesn't appear if one averages the second moment of the L -functions.

3. Since f is a self dual Hecke-Maass form of $GL(3)$, it has to be orthogonal ([JS]) which means the (partial) L -function $L^S(s, f, \text{sym}^2)$ has a pole at $s = 1$; since u_j is a Maass form of $GL(2)$, it is symplectic which means $L^S(s, u_j, \text{sym}^2)$ has no pole at $s = 1$. Then Lapid's theorem [La] says that $L(\frac{1}{2}, f \times u_j) \geq 0$. Due to this important property, we have

Corollary 1.1. *Under the same assumptions as in the above theorem,*

$$L\left(\frac{1}{2}, f \times u_j\right) \ll_{\varepsilon, f} (1 + |t_j|)^{\frac{11}{8}+\varepsilon}.$$

The corresponding convexity bound for $L(\frac{1}{2}, f \times u_j)$ is $t_j^{\frac{3}{2}+\varepsilon}$ with $\varepsilon > 0$, so the above bound breaks the convexity bound.

Remarks 1. The nonnegativity of $L(\frac{1}{2}, f \times u_j)$ plays a crucial role in our approach. Otherwise, one can hardly motivate the goal of studying the first moment.

2. In the case that f is an Eisenstein series on $GL(3)$, our approach recovers the subconvexity of a $GL(2)$ L -function in the eigenvalue aspect.

Ignoring the contribution of the cuspidal spectrum in (1.3) by the nonnegativity

of $L(\frac{1}{2}, f \times u_j)$ [La], one has

$$\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{T^{\frac{3}{4}}}} \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt \ll_{\varepsilon, f} T^{\frac{11}{8} + \varepsilon}.$$

By a standard argument [He], we have

Corollary 1.2. *For f a self dual Hecke-Maass form for $SL(3, \mathbb{Z})$,*

$$L\left(\frac{1}{2} - it, f\right) \ll_{\varepsilon, f} (|t| + 1)^{\frac{11}{16} + \varepsilon}$$

where $\varepsilon > 0$.

The corresponding convexity bound for $L(\frac{1}{2} - it, f)$ is $|t|^{\frac{3}{4} + \varepsilon}$ with $\varepsilon > 0$, so the above bound breaks the convexity bound for $L(\frac{1}{2} - it, f)$ in the t -aspect.

Remark. Our method only breaks the convexity bounds of $L(\frac{1}{2}, f \times u_j)$ and $L(\frac{1}{2}, f)$ with f self dual on $GL(3)$, i.e., f comes from the symmetric lifts from $GL(2)$ (see [So]). New ideas are needed for the more general case f is non self dual on $GL(3)$.

We end the introduction by a brief outline of the proof of the main theorem. Because we restrict to averaging over even Maass forms in (1.3), applying the approximate functional equation for the Rankin-Selberg L -functions and Kuznetsov's formula leads to two parts: $\tilde{\mathcal{R}}_3^+$ (see (4.17)) – weighted sums of Kloosterman sums twisted by $e^{\frac{4\pi i \sqrt{n}}{c}}$ and $\tilde{\mathcal{R}}_2^-$ (see (5.10)) – weighted sums of Kloosterman sums without twisting. Instead of using Weil's bound for the Kloosterman sum which only leads to the convexity bound for the individual L -function, we expand the Kloosterman sums and makes crucial use of the Voronoi formula on $GL(3)$. $\tilde{\mathcal{R}}_2^-$ involves no twisting which allows a direct application of the Voronoi formula. $\tilde{\mathcal{R}}_3^+$ seems harder. However, as a miracle, the application of the Voronoi formula to $\tilde{\mathcal{R}}_3^+$ brings the twists by $e^{\frac{4\pi i \sqrt{n}}{c}}$ to twists by additive characters (see (4.24)). This breaks the duality of the Voronoi formula. A second application of the Voronoi formula twisted by additive characters then completes the estimation of \mathcal{R}_3^+ . In using the Voronoi formula, one needs the asymptotic behavior of the integral transformations of the test functions. This is provided in Lemma 2.1. In the appendix, suggested by Sarnak, we also considered the subconvexity of the Rankin-Selberg L -function $L(s, f \times h)$ where f is self dual on $GL(3)$ and h runs through holomorphic forms of weight k congruent to 0 modulo 4. The analysis is essentially the same as the nonholomorphic case.

The Voronoi formula for $GL(3)$ was first derived by Miller and Schmidt [MS] (see [GL] for a simple proof). It was first used by Sarnak and Watson to prove a Lindeloff like bound for the L^4 norm of a Maass form for $GL(2)$. For other applications, see [Mi] and [Li]. Throughout the paper, $e(x)$ means $e^{2\pi i x}$ and negligible means $O(T^{-A})$ for any $A > 0$.

2 A review of automorphic forms

In this section, we introduce notations and recall some standard facts of Maass forms for $GL(2)$ and $GL(3)$. We start from the upper half plane \mathbb{H} . The Laplace operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

has a spectral decomposition on $L^2(SL(2, \mathbb{Z}) \backslash \mathbb{H})$:

$$L^2(SL(2, \mathbb{Z}) \backslash \mathbb{H}) = \mathcal{C} \oplus C(SL(2, \mathbb{Z}) \backslash \mathbb{H}) \oplus \mathcal{E}(SL(2, \mathbb{Z}) \backslash \mathbb{H}).$$

Here \mathcal{C} is the space of constant functions. $C(SL(2, \mathbb{Z}) \backslash \mathbb{H})$ is the space of Maass forms and $\mathcal{E}(SL(2, \mathbb{Z}) \backslash \mathbb{H})$ is the space of Eisenstein series.

Let $\mathcal{U} = \{u_j : j \geq 1\}$ be an orthonormal basis of Hecke-Maass forms corresponding to the Laplacian eigenvalue $\frac{1}{4} + t_j^2$ with $t_j \geq 0$ in the space $C(SL(2, \mathbb{Z}) \backslash \mathbb{H})$. Any $u_j(z)$ has the Fourier expansion

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) W_{s_j}(nz)$$

where $W_s(z)$ is the Whittaker function given by

$$W_s(z) = 2|y|^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|y|) e(x)$$

and $K_s(y)$ is the K -Bessel function with $s = \frac{1}{2} + it$. $C(SL(2, \mathbb{Z}) \backslash \mathbb{H})$ consists of even Maass forms and odd Maass forms according to $u_j(-\bar{z}) = u_j(z)$ or $u_j(-\bar{z}) = -u_j(z)$. We can assume u_j are eigenfunctions of all the Hecke operators corresponding to the Hecke eigenvalue $\lambda_j(n)$. Then we have the formula

$$\rho_j(\pm n) = \rho_j(\pm 1) \lambda_j(n) n^{-\frac{1}{2}}$$

if $n > 0$. The Eisenstein series $E(z, s)$ defined by

$$(2.1) \quad E(z, s) = \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) = 1}} \frac{y^s}{|cz + d|^{2s}}$$

has the following Fourier expansion

$$E(z, s) = y^s + \phi(s) y^{1-s} + \sum_{n \neq 0} \phi(n, s) W_s(nz)$$

where

$$\phi(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)}$$

with $\zeta(s)$ be the Riemann zeta function and

$$\phi(n, s) = \pi^s \Gamma(s)^{-1} \zeta(2s)^{-1} |n|^{-\frac{1}{2}} \eta(n, s)$$

with

$$(2.2) \quad \eta(n, s) = \sum_{ad=|n|} \left(\frac{a}{d}\right)^{s-\frac{1}{2}}.$$

For any $m, n \geq 1$ and any test function $h(t)$ which is even and satisfies the following conditions:

I) $h(t)$ is holomorphic in $|\Im t| \leq \frac{1}{2} + \varepsilon$;

II) $h(t) \ll (|t| + 1)^{-2-\varepsilon}$ in the above strip, we have the following Kuznetsov formula (see [CI])

$$(2.3) \quad \sum'_{j \geq 1} h(t_j) \omega_j \lambda_j(m) \lambda_j(n) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \omega(t) \bar{\eta} \left(m, \frac{1}{2} + it\right) \eta \left(n, \frac{1}{2} + it\right) dt \\ = \frac{1}{2} \delta(m, n) H + \sum_{c > 0} \frac{1}{2c} \left\{ S(m, n; c) H^+ \left(\frac{4\pi\sqrt{mn}}{c}\right) + S(-m, n; c) H^- \left(\frac{4\pi\sqrt{mn}}{c}\right) \right\}$$

where \sum' restricts to the even Maass forms, $\delta(m, n)$ is the Kronecker symbol,

$$\omega_j = 4\pi |\rho_j(1)|^2 / \cosh \pi t_j, \\ \omega(t) = 4\pi \left| \phi \left(1, \frac{1}{2} + it\right) \right|^2 \cosh^{-1} \pi t, \\ H = \frac{2}{\pi} \int_0^{\infty} h(t) \tanh(\pi t) t dt, \\ H^+(x) = 2i \int_{-\infty}^{\infty} J_{2it}(x) \frac{h(t)t}{\cosh \pi t} dt, \\ H^-(x) = \frac{4}{\pi} \int_{-\infty}^{\infty} K_{2it}(x) \sinh(\pi t) h(t) t dt,$$

$$S(a, b; c) = \sum_{d\bar{d} \equiv 1 \pmod{c}} e\left(\frac{da + \bar{d}b}{c}\right)$$

is the classical Kloosterman sum, in the above, $J_\nu(x)$ and $K_\nu(x)$ are the standard J -Bessel function and K -Bessel function respectively.

Now we recall some background on Maass forms for $GL(3)$. We will follow the notations in Goldfeld's book [Gol]. Let f be a Maass form of type $\nu = (\nu_1, \nu_2)$ for $SL(3, \mathbb{Z})$. Thanks to Jacquet, Piatetskii-Shapiro and Shalika, we have the following Fourier Whittaker expansion

$$(2.4) \quad f(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \backslash SL(2, \mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{m_1 |m_2|} W_J \left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,1} \right)$$

where $U_2(\mathbb{Z})$ is the group of 2×2 upper triangular matrices with integer entries and ones on the diagonal, $W_J(z, \nu, \psi_{1,1})$ is the Jacquet-Whittaker function and $M = \text{diag}(m_1 | m_2, m_1, 1)$. Set

$$\alpha = -\nu_1 - 2\nu_2 + 1, \quad \beta = -\nu_1 + \nu_2, \quad \gamma = 2\nu_1 + \nu_2 - 1,$$

for $k = 0, 1$; for $\psi(x)$ a smooth compactly supported function on $(0, \infty)$ and $\tilde{\psi}(s) := \int_0^\infty \psi(x) x^s \frac{dx}{x}$, set

$$(2.5) \quad \Psi_k(x) := \int_{\Re s = \sigma} (\pi^3 x)^{-s} \frac{\Gamma\left(\frac{1+s+2k+\alpha}{2}\right) \Gamma\left(\frac{1+s+2k+\beta}{2}\right) \Gamma\left(\frac{1+s+2k+\gamma}{2}\right)}{\Gamma\left(\frac{-s-\alpha}{2}\right) \Gamma\left(\frac{-s-\beta}{2}\right) \Gamma\left(\frac{-s-\gamma}{2}\right)} \tilde{\psi}(-s-k) ds$$

with $\sigma > \max\{-1 - \Re\alpha, -1 - \Re\beta, -1 - \Re\gamma\}$,

$$\Psi_{0,1}^0(x) = \Psi_0(x) + \frac{\pi^{-3} c^3 m}{n_1^2 n_2 i} \Psi_1(x)$$

and

$$\Psi_{0,1}^1(x) = \Psi_0(x) - \frac{\pi^{-3} c^3 m}{n_1^2 n_2 i} \Psi_1(x),$$

we have the following Voronoi formula on $\text{GL}(3)$:

Proposition 2.1. ([MS], [GL]) *Let $\psi(x) \in C_c^\infty(0, \infty)$. Let $A(m, n)$ denote the (m, n) -th Fourier coefficient of a Maass form for $SL(3, \mathbb{Z})$ as in (2.4). Let $d, \bar{d}, c \in \mathbb{Z}$ with $c \neq 0, (d, c) = 1$, and $d\bar{d} \equiv 1 \pmod{c}$. Then we have*

$$\begin{aligned} & \sum_{n>0} A(m, n) e\left(\frac{n\bar{d}}{c}\right) \psi(n) \\ &= \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_1 n_2} S(md, n_2; mcn_1^{-1}) \Psi_{0,1}^0\left(\frac{n_2 n_1^2}{c^3 m}\right) \\ &+ \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_1 n_2} S(md, -n_2; mcn_1^{-1}) \Psi_{0,1}^1\left(\frac{n_2 n_1^2}{c^3 m}\right), \end{aligned}$$

where $S(a, b; c)$ is the Kloosterman sum defined as the above.

To apply Proposition 2.1 in practice, one needs to know the asymptotic behaviour of $\Psi_0(x)$ and $\Psi_1(x)$. By changing variables $s+1 \rightarrow s$ in the definition of $\Psi_1(x)$, one sees that $x^{-1}\Psi_1(x)$ has similar asymptotic behavior as of $\Psi_0(x)$. Therefore, in the following, we only consider $\Psi_0(x)$.

Lemma 2.1. ([Li]) *Suppose $\psi(x)$ is a smooth function compactly supported on $[X, 2X]$, $\Psi_0(x)$ is defined by (2.5), then for any fixed integer $K \geq 1$ and $xX \gg 1$,*

we have

$$\begin{aligned} \Psi_0(x) = 2\pi^4 xi \int_0^\infty \psi(y) \sum_{j=1}^K \frac{c_j \cos(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}}) + d_j \sin(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}})}{(\pi^3 xy)^{\frac{j}{3}}} dy \\ + O\left((xX)^{-\frac{K+2}{3}}\right), \end{aligned}$$

where c_j and d_j are constants depending on α, β and γ , in particular, $c_1 = 0, d_1 = -\frac{2}{\sqrt{3\pi}}$.

Remark. When $xX \ll 1$, moving the line of integration to $\sigma = -\frac{11}{20}$, by Stirling's formula for the Γ functions and integration by part once for $\tilde{\psi}(s)$, one shows that

$$\Psi_0(x) \ll \int_0^\infty |\psi'(x)| dx.$$

Note that a special case of the above lemma (when $\alpha = \beta = \gamma = 0$) was given by Ivic (see [Iv]). Now let f be a self dual Hecke-Maass form of type (ν, ν) for $SL(3, \mathbb{Z})$, normalized to have the first Fourier coefficient $A(1, 1)$ equal to 1. We associate the L -function $L(s, f)$ defined by (1.1). It is entire and satisfies the functional equation

$$G_\nu(s)L(s, f) = G_\nu(1-s)L(1-s, f)$$

where

$$G_\nu(s) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s+1-3\nu}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-1+3\nu}{2}\right).$$

The Rankin-Selberg L -function defined by

$$L(s, f \times f) := \sum_{m \geq 1} \sum_{n \geq 1} \frac{|A(m, n)|^2}{(m^2 n)^s}$$

for $\Re s$ large has a meromorphic continuation to the whole plane with the only simple pole at $s = 1$. By a standard contour integration, one shows that

$$(2.6) \quad \sum_{m^2 n \leq N} |A(m, n)|^2 \ll_f N.$$

By Cauchy's inequality and (2.6), one derives that

$$(2.7) \quad \sum_{n \leq N} |A(m, n)| \ll_f N |m|.$$

The Rankin-Selberg L -function of f and u_j defined by (1.2) is entire and satisfies the functional equation

$$(2.8) \quad \Lambda(s, f \times u_j) = \Lambda(1-s, f \times u_j)$$

where

$$\begin{aligned} \Lambda(s, f \times u_j) &= \pi^{-3s} \Gamma\left(\frac{s - it_j - \alpha}{2}\right) \Gamma\left(\frac{s - it_j - \beta}{2}\right) \Gamma\left(\frac{s - it_j - \gamma}{2}\right) \\ &\quad \times \Gamma\left(\frac{s + it_j - \alpha}{2}\right) \Gamma\left(\frac{s + it_j - \beta}{2}\right) \Gamma\left(\frac{s + it_j - \gamma}{2}\right) L(s, f \times u_j) \end{aligned}$$

and

$$(2.9) \quad \alpha = -3\nu + 1, \quad \beta = 0, \quad \gamma = 3\nu - 1.$$

To the above Maass form f and the Eisenstein series $E(z, \frac{1}{2} + it)$ (recall (2.1)) we associate the L -function

$$L(s, f \times E) := \sum_{m \geq 1} \sum_{n \geq 1} \frac{\bar{\eta}(n, \frac{1}{2} + it) A(n, m)}{(m^2 n)^s}.$$

By looking at the Euler products

$$L(s, f) = \sum_{n \geq 1} \frac{A(n, 1)}{n^s} = \prod_p \prod_{i=1}^3 (1 - \beta_{p,i} p^{-s})^{-1},$$

$$L(s, E) = \sum_{n \geq 1} \eta\left(n, \frac{1}{2} + it\right) n^{-s} = \prod_p (1 - p^{-s+it})^{-1} (1 - p^{-s-it})^{-1},$$

one derives that (see [Gol] pp. 379)

$$\begin{aligned} L(s, f \times E) &= \prod_p \prod_{k=1}^3 (1 - \beta_{p,k} p^{it-s})^{-1} (1 - \beta_{p,k} p^{-it-s})^{-1} \\ &= L(s - it, f) L(s + it, f). \end{aligned}$$

It yields that

$$L\left(\frac{1}{2}, f \times E\right) = \left| L\left(\frac{1}{2} - it, f\right) \right|^2.$$

This satisfies the functional equation (2.8) which can also be verified directly using the functional equation of $L(s, f)$. Set

$$F(u) = \left(\cos \frac{\pi u}{A} \right)^{-3A},$$

for $|\Im t| \leq 1000$, where A is a positive integer,

$$(2.10) \quad V(y, t) = \frac{1}{2\pi i} \int_{(1000)} y^{-u} F(u) \frac{\gamma(\frac{1}{2} + u, t)}{\gamma(\frac{1}{2}, t)} \frac{du}{u}$$

and

$$\begin{aligned} \gamma(s, t) = \pi^{-3s} \Gamma\left(\frac{s - it - \alpha}{2}\right) \Gamma\left(\frac{s - it - \beta}{2}\right) \Gamma\left(\frac{s - it - \gamma}{2}\right) \\ \times \Gamma\left(\frac{s + it - \alpha}{2}\right) \Gamma\left(\frac{s + it - \beta}{2}\right) \Gamma\left(\frac{s + it - \gamma}{2}\right). \end{aligned}$$

The integral is justified by Luo-Rudnick-Sarnak's bound on the Ramanujan conjecture $|\Re \alpha|, |\Re \beta|, |\Re \gamma| \leq \frac{1}{2} - \frac{1}{10}$ (see [LRS]). One has the following approximate functional equation for $L(s, f \times u_j)$ (see [IK] or [Li]):

Lemma 2.2. *For a self dual Maass form f of type (ν, ν) for $SL(3, \mathbb{Z})$ and any $u_j(z)$ associated to the Laplacian eigenvalue $\frac{1}{4} + t_j^2$ in the orthonormal basis of even Hecke-Maass forms for $SL(2, \mathbb{Z})$, we have*

$$(2.11) \quad L\left(\frac{1}{2}, f \times u_j\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(n, m)}{(m^2 n)^{\frac{1}{2}}} V(m^2 n, t_j).$$

$V(y, t)$ has the following properties which effectively limit the terms in (2.11) with $m^2 n \ll |t_j|^3$.

Lemma 2.3. *For $y, t > 0, i = 1, 2$,*

1) *the derivatives of $V(y, t)$ with respect to y satisfy*

$$\begin{aligned} y^a \frac{\partial^a}{\partial y^a} V(y, t) &\ll \left(1 + \frac{y}{|t|^3}\right)^{-A}, \\ y^a \frac{\partial^a}{\partial y^a} V(y, t) &= \delta_a + O\left(\left(\frac{y}{|t|^3}\right)^c\right), \end{aligned}$$

where $0 < c \leq \frac{1}{3} \min\{\frac{1}{2} - \Re \alpha, \frac{1}{2} - \Re \beta, \frac{1}{2} - \Re \gamma\}$, $\delta_0 = 1, 0$ otherwise and the implied constants depend only on c, a, A, α, β and γ .

2) *if $1 \leq y \ll t^{3+\varepsilon}$, then as $t \rightarrow \infty$, we have*

$$\begin{aligned} V(y, t) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left(\frac{t^3}{8\pi^3 y}\right)^u F(u) \left[1 + \frac{p_1(v)}{t} + \dots + \frac{p_{n-1}(v)}{t^{n-1}} + O\left(\frac{p_n(v)}{t^n}\right)\right] \frac{du}{u} \\ + O(t^{-B}) \end{aligned}$$

where $v = \Im u$, $p_i(v)$ are polynomials of v and B is arbitrarily large.

Proof. 1) See [IK], pp. 100.

2) It follows from Stirling's formula

$$\log \Gamma(s + b) = \left(s + b - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \sum_{j=1}^k \frac{a_j}{s^j} + O_\delta\left(\frac{1}{|s|^{k+1}}\right),$$

which is valid for b a constant, any fixed integer $K \geq 1, |\arg s| \leq \pi - \delta$ for $\delta > 0$, where the point $s = 0$ and the neighbourhoods of the poles of $\Gamma(s + b)$

are excluded, and the a_j are suitable constants. \square

$L(s, f \times E)$ has the similar approximate functional equation as the above

$$(2.12) \quad L\left(\frac{1}{2}, f \times E\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\eta(n, \frac{1}{2} + it) A(n, m)}{(m^2 n)^{\frac{1}{2}}} V(m^2 n, t).$$

Now we introduce the spectrally normalized first moment of the central values of L -functions

$$(2.13) \quad W := \sum_j' e^{-\frac{(t_j - T)^2}{M^2}} \omega_j L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-\frac{(t - T)^2}{M^2}} \omega(t) \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt$$

where ω_j and $\omega(t)$ are defined below (2.3). Due to Iwaniec [Iw2], we know

$$\omega_j \gg t_j^{-\varepsilon}$$

and as a well-known fact ([Ti], pp. 111) we also know

$$\omega(t) \gg t^{-\varepsilon},$$

one has

$$\sum_j' e^{-\frac{(t_j - T)^2}{M^2}} L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-\frac{(t - T)^2}{M^2}} \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt \ll WT^\varepsilon$$

for any $\varepsilon > 0$. Therefore, for Theorem 1.1 we need to show that

$$(2.14) \quad W \ll_{\varepsilon, f} T^{1+\varepsilon} M.$$

To use the Kuznetsov formula, the test function has to be even. For that purpose, we introduce

$$(2.15) \quad \mathcal{W} := \sum_j' k(t_j) \omega_j L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} k(t) \omega(t) \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt,$$

here

$$(2.16) \quad k(t) = e^{-\frac{(t - T)^2}{M^2}} + e^{-\frac{(t + T)^2}{M^2}}.$$

Applying (2.11) and (2.12) to \mathcal{W} , by smooth dyadic subdivisions it suffices for our purposes to estimate sums of the form

$$(2.17) \quad \mathcal{R} := 2 \sum_j' k(t_j) \omega_j \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(n, m)}{(m^2 n)^{\frac{1}{2}}} V(m^2 n, t_j) g\left(\frac{m^2 n}{N}\right) \\ + \frac{2}{4\pi} \int_{-\infty}^{\infty} k(t) \omega(t) \sum_{m \geq 1} \sum_{n \geq 1} \frac{\eta(n, \frac{1}{2} + it) A(n, m)}{(m^2 n)^{\frac{1}{2}}} V(m^2 n, t) g\left(\frac{m^2 n}{N}\right) dt.$$

Here g is essentially a fixed smooth function of compact support on $[1, 2]$ and N is at most $T^{3+\varepsilon}$, $\varepsilon > 0$. We then transform \mathcal{R} by the Kuznetsov formula (2.3) into

$$(2.18) \quad \mathcal{R} = \mathcal{D} + \mathcal{R}^+ + \mathcal{R}^-$$

where

$$(2.19) \quad \mathcal{D} = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \delta(n, 1) H_{m, n}$$

is the contribution of the diagonal term with

$$(2.20) \quad H_{m, n} = \frac{2}{\pi} \int_0^\infty k(t) V(m^2 n, t) \tanh(\pi t) t dt,$$

$$(2.21) \quad \mathcal{R}^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c > 0} c^{-1} S(n, 1; c) H_{m, n}^+ \left(\frac{4\pi\sqrt{n}}{c}\right)$$

with

$$(2.22) \quad H_{m, n}^+(x) = 2i \int_{-\infty}^\infty J_{2it}(x) \frac{k(t) V(m^2 n, t) t}{\cosh \pi t} dt$$

and

$$(2.23) \quad \mathcal{R}^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c > 0} c^{-1} S(n, 1; c) H_{m, n}^- \left(\frac{4\pi\sqrt{n}}{c}\right)$$

with

$$(2.24) \quad H_{m, n}^-(x) = \frac{4}{\pi} \int_{-\infty}^\infty K_{2it}(x) \sinh(\pi t) k(t) V(m^2 n, t) t dt.$$

The next three sections are devoted to the estimation of \mathcal{D} , \mathcal{R}^+ , and \mathcal{R}^- respectively.

3 The diagonal terms

Recall that \mathcal{D} is the contribution to \mathcal{R} (see (2.18)) from the diagonal terms defined by (2.19). Obviously

$$\mathcal{D} = \sum_{m \geq 1} \frac{A(1, m)}{m} g\left(\frac{m^2}{N}\right) H_{m, 1}$$

where

$$\begin{aligned}
 (3.1) \quad H_{m,1} &= \frac{2}{\pi} \int_0^\infty \left[e^{-\frac{(t-T)^2}{M^2}} + e^{-\frac{(t+T)^2}{M^2}} \right] V(m^2, t) \tanh(\pi t) t dt \\
 &= \frac{2}{\pi} \int_0^\infty e^{-\frac{(t-T)^2}{M^2}} V(m^2, t) \tanh(\pi t) t dt + O(T^{-A})
 \end{aligned}$$

with A arbitrarily large. By Lemma 2.3 and (2.7), we have

$$(3.2) \quad \sum_{m \geq 1} \frac{A(1, m)}{m} g\left(\frac{m^2}{N}\right) V(m^2, t) \ll_{\varepsilon, f} (|t| + 1)^\varepsilon.$$

It follows from (3.1) and (3.2) that

$$\mathcal{D} \ll_{\varepsilon, f} T^{1+\varepsilon} M$$

as we want.

4 The terms related to the J -Bessel function

This section is devoted to the estimation of \mathcal{R}^+ which is defined by (2.21).

We split \mathcal{R}^+ into three parts $\mathcal{R}_1^+, \mathcal{R}_2^+, \mathcal{R}_3^+$ with

$$(4.1) \quad \mathcal{R}_1^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \geq C_1/m} c^{-1} S(n, 1; c) H_{m,n}^+ \left(\frac{4\pi\sqrt{n}}{c}\right),$$

$$(4.2) \quad \mathcal{R}_2^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{C_2/m \leq c \leq C_1/m} c^{-1} S(n, 1; c) H_{m,n}^+ \left(\frac{4\pi\sqrt{n}}{c}\right),$$

$$(4.3) \quad \mathcal{R}_3^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \leq C_2/m} c^{-1} S(n, 1; c) H_{m,n}^+ \left(\frac{4\pi\sqrt{n}}{c}\right)$$

where

$$(4.4) \quad C_1 = T, \quad C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon} M}.$$

First we will estimate (4.1). Recall $H_{m,n}^+(x)$ is defined by (2.22). Moving the line of integration to $\Im t = -100$, $H_{m,n}^+(x)$ becomes

$$(4.5) \quad 2i \int_{-\infty}^{\infty} J_{2iy+200}(x) \frac{k(-100i+y) V(m^2 n, -100i+y) (-100i+y)}{\cosh \pi(-100i+y)} dy.$$

By the integral representation of the J -Bessel function ([GR], 8.411 4)

$$J_\nu(z) = 2 \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} \sin^{2\nu} \theta \cos(z \cos \theta) d\theta$$

for $\Re \nu > -\frac{1}{2}$, one derives that

$$(4.6) \quad J_{2iy+200}(x) \ll \left(\frac{x}{|y|}\right)^{200} e^{\pi|y|}.$$

Using Stirling's formula, we have

$$(4.7) \quad V(m^2 n, -100i + y) \ll \left(\frac{|y|^3}{m^2 n}\right)^{100}.$$

Combining (4.5), (4.6) and (4.7), we have

$$(4.8) \quad H_{m,n}^+(x) \ll x^{200} T^{100} (m^2 n)^{-100} T M.$$

Thus, by (2.7), (4.8) and the trivial bound for the Kloosterman sum, one concludes that

$$(4.9) \quad \mathcal{R}_1^+ \ll N^{\frac{1}{2}} T^{-98} M \ll 1.$$

Next we will estimate \mathcal{R}_2^+ . By [GR] (8.411 11), one derives that

$$\frac{J_{2it}(x) + J_{-2it}(x)}{\cosh \pi t} = \frac{2}{\pi} \int_{-\infty}^{\infty} \sin(x \cosh \zeta) e\left(\frac{t\zeta}{\pi}\right) d\zeta.$$

Applying the above integral representation and partial integration in ζ once, we have

$$H_{m,n}^+(x) = \frac{4i}{\pi} \int_{t=0}^{\infty} \int_{\zeta=-T^\varepsilon}^{T^\varepsilon} t e^{-\frac{(t-T)^2}{M^2}} V(m^2 n, t) \sin(x \cosh \zeta) e\left(\frac{t\zeta}{\pi}\right) dt d\zeta \\ + O(T^{-A})$$

with A arbitrarily large. By changing variables $\frac{t-T}{M} \rightarrow t$, we have

$$H_{m,n}^+(x) = \frac{4iM}{\pi} \int_{t=-\frac{T}{M}}^{\infty} \int_{\zeta=-T^\varepsilon}^{T^\varepsilon} (T + tM) e^{-t^2} V(m^2 n, tM + T) \sin(x \cosh \zeta) \\ \times e\left(\frac{(tM + T)\zeta}{\pi}\right) dt d\zeta + O(T^{-A}).$$

Extending the t integral to $(-\infty, \infty)$ with a negligible error term, we have

$$H_{m,n}^+(x) = H_{m,n}^{+,1}(x) + H_{m,n}^{+,2}(x) + O(T^{-A})$$

where

$$H_{m,n}^{+,1}(x) = \frac{4iMT}{\pi} \int_{t=-\infty}^{\infty} \int_{\zeta=-T^\varepsilon}^{\zeta=T^\varepsilon} e^{-t^2} V(m^2n, tM + T) \sin(x \cosh \zeta) e\left(\frac{tM\zeta}{\pi}\right) \times e\left(\frac{T\zeta}{\pi}\right) dt d\zeta$$

and

$$H_{m,n}^{+,2}(x) = \frac{4iM^2}{\pi} \int_{t=-\infty}^{\infty} \int_{\zeta=-T^\varepsilon}^{T^\varepsilon} te^{-t^2} V(m^2n, tM + T) \sin(x \cosh \zeta) e\left(\frac{tM\zeta}{\pi}\right) \times e\left(\frac{T\zeta}{\pi}\right) dt d\zeta.$$

In the following we only treat $H_{m,n}^{+,1}(x)$ since $H_{m,n}^{+,2}(x)$ is a lower order term which can be handled in a similar way. It is clear that

$$(4.10) \quad H_{m,n}^{+,1}(x) = \frac{4iMT}{\pi} \int_{\zeta=-T^\varepsilon}^{T^\varepsilon} \hat{k}^*\left(-\frac{M\zeta}{\pi}\right) \sin(x \cosh \zeta) e\left(\frac{T\zeta}{\pi}\right) d\zeta$$

which is equal to

$$4iT \int_{\zeta=-\frac{MT^\varepsilon}{\pi}}^{\frac{MT^\varepsilon}{\pi}} \hat{k}^*(\zeta) \sin\left(x \cosh \frac{\zeta\pi}{M}\right) e\left(-\frac{T\zeta}{M}\right) d\zeta$$

by making a change of variable $-\frac{M\zeta}{\pi} \rightarrow \zeta$, here

$$(4.11) \quad k^*(t) = e^{-t^2} V(m^2n, tM + T)$$

and

$$(4.12) \quad \hat{k}^*(\zeta) = \int_{-\infty}^{\infty} k^*(t) e(-t\zeta) dt$$

is its Fourier transform. Since $\hat{k}^*(\zeta)$ is a Schwartz class function, one can extend the integral in (4.10) to $(-\infty, \infty)$ with a negligible error term. Now let

$$(4.13) \quad W_{m,n}(x) := T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) \sin\left(x \cosh \frac{\zeta\pi}{M}\right) e\left(-\frac{T\zeta}{M}\right) d\zeta$$

and

$$(4.14) \quad W_{m,n}^*(x) := T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) e \left(-\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh \frac{\zeta\pi}{M} \right) d\zeta,$$

then

$$W_{m,n}(x) = \frac{W_{m,n}^*(-x) - W_{m,n}^*(x)}{2i}$$

and

$$H_{m,n}^{+,1}(x) = 4iW_{m,n}(x) + O(T^{-A})$$

with A arbitrarily large. The contribution to $W_{m,n}(x)$ coming from $|\zeta| \geq T^\varepsilon$ ($\varepsilon > 0$ arbitrarily small but fixed) is negligible. So we need only consider $|\zeta| \leq T^\varepsilon$. The phase ϕ in the exponential of $W_{m,n}^*(x)$ is

$$\phi(\zeta) = -\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh \frac{\zeta\pi}{M},$$

so

$$\phi'(\zeta) = -\frac{T}{M} - \frac{x}{2M} \sinh \frac{\zeta\pi}{M}.$$

Then if $|x| \leq T^{1-\varepsilon}M$, $W_{m,n}^*(x)$ is negligible. In the following we assume that

$$T^{1-\varepsilon}M \leq |x| \leq M^4.$$

In this case we need the asymptotic expansion of $W_{m,n}^*(x)$. One could quote Lemma 5.1 of [LYL]. For completeness, we prefer to derive it here. But the methods are really based on [Sa] and [LYL]. Now

$$\begin{aligned} W_{m,n}^*(x) = T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) e \left(-\frac{T\zeta}{M} - \frac{x}{2\pi} - \frac{\pi x \zeta^2}{4M^2} - \frac{\pi^3 x \zeta^4}{48M^4} - \frac{\pi^5 x \zeta^6}{1440M^6} \right) d\zeta \\ + O \left(T \int_{-\infty}^{\infty} |\hat{k}^*(\zeta)| \frac{|\zeta|^8 |x|}{M^8} d\zeta \right). \end{aligned}$$

Expanding $e \left(\frac{-\pi^5 x \zeta^6}{1440M^6} \right)$ into a Taylor series of order 1, we have

$$(4.15) \quad W_{m,n}^*(x) = W_{m,n}^+(x) - \frac{2\pi^6 i x}{1440M^6} W_{m,n}^-(x) + O \left(\frac{T|x|}{M^8} \right),$$

where

$$W_{m,n}^+(x) = T e \left(\frac{-x}{2\pi} \right) \int_{-\infty}^{\infty} k_0^*(\zeta) e \left(-\frac{T\zeta}{M} - \frac{\pi x \zeta^2}{4M^2} \right) d\zeta$$

with

$$k_0^*(\zeta) = \hat{k}^*(\zeta) e \left(\frac{-\pi^3 x \zeta^4}{48M^4} \right)$$

and

$$W_{m,n}^-(x) = Te \left(\frac{-x}{2\pi} \right) \int_{-\infty}^{\infty} k_1^*(\zeta) e \left(-\frac{T\zeta}{M} - \frac{\pi x \zeta^2}{4M^2} \right) d\zeta$$

with

$$k_1^*(\zeta) = \zeta^6 \hat{k}^*(\zeta) e \left(\frac{-\pi^3 x \zeta^4}{48M^4} \right).$$

Now by completing the square, we have

$$W_{m,n}^+(x) = Te \left(\frac{-x}{2\pi} + \frac{T^2}{\pi x} \right) \int_{-\infty}^{\infty} k_0^*(\zeta) e \left(-\frac{\pi x}{4M^2} \left(\zeta + \frac{2MT}{\pi x} \right)^2 \right) d\zeta$$

which is equal to ([GR], 3.691 1)

$$(1+i)Te \left(\frac{-x}{2\pi} + \frac{T^2}{\pi x} \right) \int_{-\infty}^{\infty} \hat{k}_0^*(\zeta) e \left(\frac{-2MT\zeta}{\pi x} \right) \frac{M}{\sqrt{\pi|x|}} e \left(\frac{M^2\zeta^2}{\pi x} \right) d\zeta$$

by Parseval. Expanding $e \left(\frac{M^2\zeta^2}{\pi x} \right)$ in a Taylor series we have

$$\begin{aligned} W_{m,n}^+(x) &= (1+i) \frac{TM}{\sqrt{\pi|x|}} e \left(\frac{-x}{2\pi} + \frac{T^2}{\pi x} \right) \\ &\quad \times \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{2iM^2}{x} \right)^l \int_{-\infty}^{\infty} \zeta^{2l} \hat{k}_0^*(\zeta) e \left(\frac{-2MT\zeta}{\pi x} \right) d\zeta \\ &= (1+i) \frac{TM}{\sqrt{\pi|x|}} e \left(\frac{-x}{2\pi} + \frac{T^2}{\pi x} \right) \sum_{l=0}^{\infty} \frac{(2i)^{-l}}{l!} \left(\frac{M^2}{\pi^2 x} \right)^l k_0^{*(2l)} \left(\frac{-2MT}{\pi x} \right). \end{aligned}$$

Since

$$k_0^{*(2l)}(t) = \sum_{0 \leq l_1 \leq 2l} \binom{2l}{l_1} \frac{d^{l_1}}{dt^{l_1}} e \left(\frac{-\pi^3 x t^4}{48M^4} \right) \times \frac{d^{2l-l_1}}{dt^{2l-l_1}} \hat{k}^*(t)$$

where $\binom{n}{r}$ denotes the binomial coefficient and

$$\left. \frac{d^{l_1}}{dt^{l_1}} e \left(\frac{-\pi^3 x t^4}{48M^4} \right) \right|_{t=\frac{-2MT}{\pi x}} \ll 1,$$

one can truncate the above series of $W_{m,n}(x)$ at order L_1 with a reminder $O \left(T \left(\frac{M}{\sqrt{x}} \right)^{2L_1+3} \right)$. Now expanding $e \left(\frac{-\pi^3 x t^4}{48M^4} \right)$ in a power series and differen-

tiating it termwisely, we have

$$\begin{aligned} \frac{d^{l_1}}{dt^{l_1}} e\left(\frac{-\pi^3 x t^4}{48M^4}\right) \Big|_{t=\frac{-2MT}{\pi x}} &= \sum_{4l_2 \geq l_1} \frac{(4l_2)!}{(4l_2 - l_1)!l_2!} \left(\frac{2i\pi^4}{48}\right)^{l_2} \left(\frac{-x}{M^4}\right)^{l_2} t^{4l_2 - l_1} \Big|_{t=\frac{-2MT}{\pi x}} \\ &= \sum_{\frac{l_1}{4} \leq l_2 \leq L_2} \frac{(4l_2)!}{(4l_2 - l_1)!l_2!} \left(\frac{i\pi^4}{24}\right)^{l_2} \left(\frac{-x}{M^4}\right)^{l_2} \left(\frac{-2MT}{\pi x}\right)^{4l_2 - l_1} \\ &\quad + O\left(\left(\frac{T^4}{|x|^3}\right)^{L_2+1} \left(\frac{|x|}{MT}\right)^{l_1}\right). \end{aligned}$$

Combining the above, we have the following asymptotic expansion

$$\begin{aligned} W_{m,n}^+(x) &= \frac{TM}{\sqrt{|x|}} e\left(\frac{-x}{2\pi} + \frac{T^2}{\pi x}\right) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{\frac{l_1}{4} \leq l_2 \leq L_2} c_{l,l_1,l_2} \\ &\quad \times \frac{M^{2l-l_1} T^{4l_2-l_1}}{x^{l+3l_2-l_1}} \hat{k}^{*(2l-l_1)} \left(\frac{-2MT}{\pi x}\right) \\ &\quad + O\left(\frac{TM}{\sqrt{|x|}} \left(\frac{T^4}{|x|^3}\right)^{L_2+1} + T \left(\frac{M}{\sqrt{|x|}}\right)^{2L_1+3}\right), \end{aligned}$$

here c_{l,l_1,l_2} are constants depending only on l, l_1 and l_2 . $W_{m,n}^-(x)$ has similar asymptotic expansion. We end up with the following proposition (recall (4.15)):

Proposition 4.1. 1) For $|x| \leq T^{1-\varepsilon}M$ with $\varepsilon > 0$,

$$W_{m,n}^*(x) \ll T^{-A}$$

where $A > 0$ is arbitrarily large and the implied constant depends on ε and A .

2) For $T^{1-\varepsilon}M \leq |x| \leq M^4$, $T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}$ and $L_2, L_1 \geq 1$,

$$\begin{aligned} (4.16) \quad W_{m,n}^*(x) &= \frac{TM}{\sqrt{|x|}} e\left(\frac{-x}{2\pi} + \frac{T^2}{\pi x}\right) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{\frac{l_1}{4} \leq l_2 \leq L_2} c_{l,l_1,l_2} \frac{M^{2l-l_1} T^{4l_2-l_1}}{x^{l+3l_2-l_1}} \\ &\quad \times \left[\hat{k}^{*(2l-l_1)} \left(\frac{-2MT}{\pi x}\right) - \frac{2\pi^6 i x}{1440M^6} (y^6 \hat{k}^*(y))^{(2l-l_1)} \left(\frac{-2MT}{\pi x}\right) \right] \\ &\quad + O\left(\frac{TM}{\sqrt{|x|}} \left(\frac{T^4}{|x|^3}\right)^{L_2+1} + T \left(\frac{M}{\sqrt{|x|}}\right)^{2L_1+3} + \frac{T|x|}{M^8}\right) \end{aligned}$$

where c_{l,l_1,l_2} are constants depending only on l, l_1 and l_2 , especially $c_{0,0,0} = \frac{1+i}{\sqrt{\pi}}$.

It follows from 1) in the above proposition, \mathcal{R}_2^+ is negligible. The remaining part of this section is devoted to the estimation of \mathcal{R}_3^+ . Applying the asymptotic expansion (4.16) of $W_{m,n}^*(x)$ and choosing L_2 and L_1 sufficiently large makes

the contribution to \mathcal{R}_3^+ from the first two terms in the error term in (4.16) negligible. The contribution to \mathcal{R}_3^+ from the last term in the error term in (4.16) is

$$O_{\varepsilon,f} \left(\frac{T^{1+\varepsilon} N}{M^8} \right) = O_{\varepsilon,f} (T^{1+\varepsilon} M)$$

as expected, where we used the trivial bound for the Kloosterman sum and (2.7). Since $|x| \geq T^{1-\varepsilon} M$,

$$\frac{M^{2l-l_1} T^{4l_2-l_1}}{x^{l-l_1+3l_2}} \ll \left(\frac{M}{T^{1-\varepsilon}} \right)^l \left(\frac{T}{M^3} \right)^{l_2} T^{(3l_2-l_1)\varepsilon} \ll 1.$$

From now on, we only take the leading term $l = 0, l_1 = 0$ and $l_2 = 0$ in (4.16). The other terms are of an identical form and can be treated similarly. We are led to estimate

$$(4.17) \quad \tilde{\mathcal{R}}_3^+ := \sqrt{2}i\pi^{-1}MT e \left(-\frac{1}{8} \right) \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{mn^{\frac{3}{4}}} g \left(\frac{m^2 n}{N} \right) \\ \times \sum_{c \leq C_2/m} c^{-\frac{1}{2}} S(n, 1; c) e \left(\frac{2\sqrt{n}}{c} - \frac{T^2 c}{4\pi^2 \sqrt{n}} \right) \hat{k}^* \left(\frac{MTc}{2\pi^2 \sqrt{n}} \right).$$

In the above, if we sum over n trivially and applying Weil's bound for the Kloosterman sum

$$S(n, 1; c) \ll_{\varepsilon} c^{\frac{1}{2}+\varepsilon},$$

we have

$$\tilde{\mathcal{R}}_3^+ \ll MTC_2^{1+\varepsilon} N^{\frac{1}{4}} \ll T^{\frac{9}{4}+\varepsilon}.$$

To save $T^{\frac{5}{4}}M^{-1}$, we have to sum over n nontrivially by the Voronoi formula for $GL(3)$ (i.e, Proposition 2.1). Expanding the Kloosterman sum in (4.17) and applying Proposition 2.1 with

$$\psi(y) = y^{-\frac{3}{4}} g \left(\frac{m^2 y}{N} \right) e \left(\frac{2\sqrt{y}}{c} - \frac{T^2 c}{4\pi^2 \sqrt{y}} \right) \hat{k}^* \left(\frac{MTc}{2\pi^2 \sqrt{y}} \right),$$

we have

$$\sum_{n \geq 1} A(n, m) e \left(\frac{nd}{c} \right) \psi(n) \\ = \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_1 n_2} S(md, n_2; mc n_1^{-1}) \Psi_{0,1}^0 \left(\frac{n_2 n_1^2}{c^3 m} \right) \\ + \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_1 n_2} S(md, -n_2; mc n_1^{-1}) \Psi_{0,1}^1 \left(\frac{n_2 n_1^2}{c^3 m} \right)$$

where $\Psi_{0,1}^0(x)$ and $\Psi_{0,1}^1(x)$ are defined below (2.5). As we explained before Proposition 2.1, we only consider the first term involving $\Psi_0(x)$ on the right

side of the above formula since all the other terms can be treated in a similar way. Since $c \leq \frac{N^{\frac{1}{2}}}{T^{1-\varepsilon} M m}$,

$$\frac{n_2 n_1^2}{c^3 m} \frac{N}{m^2} \gg T^{\frac{3}{2}},$$

by Lemma 2.1 for $x = \frac{n_2 n_1^2}{c^3 m}$,

$$(4.18) \quad \begin{aligned} \Psi_0(x) &= 2\pi^4 x i \int_0^\infty \psi(y) \frac{d_1 \sin(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}})}{(\pi^3 x y)^{\frac{1}{3}}} dy + \text{lower order terms} \\ &= \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty e(u_1(y)) a(y) dy - \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty e(u_2(y)) a(y) dy \\ &\quad + \text{lower order terms} \end{aligned}$$

where

$$\begin{aligned} u_1(y) &= \frac{2\sqrt{y}}{c} + 3x^{\frac{1}{3}} y^{\frac{1}{3}}, \\ u_2(y) &= \frac{2\sqrt{y}}{c} - 3x^{\frac{1}{3}} y^{\frac{1}{3}} \end{aligned}$$

and

$$a(y) = g\left(\frac{m^2 y}{N}\right) \hat{k}^* \left(\frac{M T c}{2\pi^2 \sqrt{y}}\right) e\left(-\frac{T^2 c}{4\pi^2 \sqrt{y}}\right) y^{-\frac{13}{12}}.$$

Since $u_1'(y) \gg c^{-1} y^{-\frac{1}{2}}$ and $a'(y) \ll T^2 c y^{-\frac{31}{12}}$, we have

$$u_1'(y) a'(y)^{-1} \gg M^2 T^{-\varepsilon} \gg T^{\frac{3}{4}-\varepsilon}.$$

By partial integration many times, one shows that the contribution to (4.17) from the first integral in (4.18) is negligible.

Now we turn to the second integral in (4.18). Since

$$u_2'(y) = \frac{1}{c} \sqrt{\frac{1}{y}} - x^{\frac{1}{3}} y^{-\frac{2}{3}},$$

if

$$(4.19) \quad x \geq 2 \frac{N^{\frac{1}{2}}}{m c^3} \quad \text{or} \quad x \leq \frac{2}{3} \frac{N^{\frac{1}{2}}}{m c^3},$$

then

$$u_2'(y) \gg \frac{1}{c} \sqrt{\frac{1}{y}}.$$

As the argument above, under the condition (4.19), the contribution to (4.17) from the second integral in (4.18) is also negligible. So for stationary or small values of $u_2'(y)$ we need only consider the case when

$$(4.20) \quad \frac{2}{3} \frac{N^{\frac{1}{2}}}{m c^3} \leq x \leq 2 \frac{N^{\frac{1}{2}}}{m c^3}, \quad \text{i.e.,} \quad \frac{2}{3} \frac{N^{\frac{1}{2}}}{n_1^2} \leq n_2 \leq 2 \frac{N^{\frac{1}{2}}}{n_1^2}.$$

Then

$$\int_0^{\infty} e(u_2(y))a(y)dy = \int_{\frac{x^{2.6}}{4}}^{\frac{9}{2}x^2c^6} e(u_2(y))a(y)dy.$$

There is a stationary phase point $y_0 = x^2c^6$ such that $u_2'(y_0) = 0$. Applying the stationary phase method ([Hu], p. 114), we have

$$(4.21) \quad \int_0^{\infty} e(u_2(y))a(y)dy = \frac{e\left(-xc^2 + \frac{1}{8}\right)a(y_0)}{\sqrt{u_2''(y_0)}} + O\left(c^{\frac{7}{2}}T^4N^{-\frac{11}{6}}m^{\frac{11}{3}}\right).$$

Due to

$$(4.22) \quad \sum_{\substack{0 \leq d < c \\ (d,c)=1}} e\left(\frac{d}{c}\right) S(md, n_2; mcn_1^{-1}) = \sum_{\substack{u \pmod{mcn_1^{-1}} \\ u\bar{u} \equiv 1 \pmod{mcn_1^{-1}}}} S(0, 1 + un_1; c) e\left(\frac{n_2\bar{u}}{mcn_1^{-1}}\right)$$

where

$$S(0, a; c) = \sum_{\substack{v \pmod{c} \\ (v,c)=1}} e\left(\frac{av}{c}\right)$$

is the Ramanujan sum which is bounded by (a, c) , we deduce that (4.22) is bounded by $mc^{1+\varepsilon}$ with $\varepsilon > 0$. Therefore, the contribution to (4.17) from the error term in (4.21) is bounded by

$$(4.23) \quad MT \sum_{m \geq 1} m^{-1} \sum_{c \leq C_2/m} c^{\frac{1}{2}} \sum_{n_1 | cm} \sum_{\substack{\frac{2N}{3n_1^2} \leq n_2 \leq 2\frac{N}{n_1^2}}} \frac{|A(n_1, n_2)|}{n_1 n_2} \\ \times \left(\frac{n_2 n_1^2}{c^3 m}\right)^{\frac{2}{3}} (mc)^{1+\varepsilon} c^{\frac{7}{2}} T^4 N^{-\frac{11}{6}} m^{\frac{11}{3}} \\ \ll M^{-3} T^{1+\varepsilon} N^{\frac{1}{2}} \ll T^{1+\varepsilon} M$$

because $M \geq T^{\frac{3}{8}}$. We conclude from (4.17), (4.21), (4.22) and (4.23) that

$$(4.24) \quad \tilde{\mathcal{R}}_3^+ = \pi^{-1} MT \sum_{m \geq 1} m^{-1} \sum_{c \leq C_2/m} c^{-1} \sum_{n_1 | cm} n_1^{-1} \sum_{n_2 > 0} A(n_1, n_2) \\ \times \sum_{\substack{0 \leq u < mcn_1^{-1} \\ u\bar{u} \equiv 1 \pmod{mcn_1^{-1}}}} S(0, 1 + un_1; c) e\left(\frac{n_2\bar{u}}{mcn_1^{-1}}\right) e\left(\frac{-n_2 n_1^2}{cm}\right) b(n_2) \\ + O(T^{1+\varepsilon} M)$$

where

$$b(y) = y^{-1} g\left(\frac{y^2 n_1^4}{N}\right) \hat{k}^* \left(\frac{MTcm}{2\pi^2 y n_1^2}\right) e\left(\frac{-T^2 cm}{4\pi^2 y n_1^2}\right).$$

If we sum over n_2 trivially, we have

$$\mathcal{R}_3^{+,*} \ll MT^{1+\varepsilon} C_2 \ll MT^{1+\varepsilon} \frac{T^{\frac{1}{2}}}{M}.$$

In order to save $T^{\frac{1}{2}} M^{-1}$, we have to sum over n_2 nontrivially using the Voronoi formula for $GL(3)$ the second time. Invoking Proposition 2.1, one has

$$(4.25) \sum_{n_2 \geq 1} A(n_1, n_2) e\left(\frac{n_2(\bar{u} - n_1)}{mc n_1^{-1}}\right) b(n_2) \\ = \frac{c' \pi^{-\frac{5}{2}}}{4i} \sum_{l_1 | c' n_1} \sum_{l_2 > 0} \frac{A(l_2, l_1)}{l_1 l_2} S(n_1 \bar{u}', l_2; n_1 c' l_1^{-1}) B_{0,1}^0 \left(\frac{l_2 l_1^2}{c'^3 n_1}\right) \\ + \frac{c' \pi^{-\frac{5}{2}}}{4i} \sum_{l_1 | c' n_1} \sum_{l_2 > 0} \frac{A(l_2, l_1)}{l_1 l_2} S(n_1 \bar{u}', -l_2; n_1 c' l_1^{-1}) B_{0,1}^1 \left(\frac{l_2 l_1^2}{c'^3 n_1}\right)$$

where

$$\frac{\bar{u} - n_1}{mc n_1^{-1}} := \frac{u'}{c'}$$

with $(u', c') = 1, c' | mc n_1^{-1}$ and $B_{0,1}^0(x)$ and $B_{0,1}^1(x)$ are defined below (2.5). As before, we only consider the first term involving $B_{0,1}^0(x)$ in (4.25) since all the other terms can be treated in a similar way. Since

$$\frac{l_2 l_1^2}{c'^3 n_1} \frac{\sqrt{N}}{n_1^2} \gg T^{1-\varepsilon} M,$$

by Lemma 2.1 for $x = \frac{l_2 l_1^2}{c'^3 n_1}$,

$$(4.26) \quad B_0(x) = 2\pi^4 x i \int_0^\infty b(y) \frac{d_1 \sin(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}})}{(\pi^3 x y)^{\frac{1}{3}}} dy + \text{lower order terms} \\ = \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty e(v_1(y)) q(y) dy - \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty e(v_2(y)) q(y) dy \\ + \text{lower order terms}$$

where

$$(4.27) \quad v_1(y) = 3x^{\frac{1}{3}} y^{\frac{1}{3}} - \frac{T^2 cm}{4\pi^2 y n_1^2},$$

$$(4.28) \quad v_2(y) = -3x^{\frac{1}{3}}y^{\frac{1}{3}} - \frac{T^2cm}{4\pi^2yn_1^2},$$

and

$$(4.29) \quad q(y) = y^{-\frac{4}{3}}g\left(\frac{y^2n_1^4}{N}\right)k^*\left(\frac{MTcm}{2\pi^2yn_1^2}\right).$$

Since

$$v_1'(y) = x^{\frac{1}{3}}y^{-\frac{2}{3}} + \frac{T^2cm}{4\pi^2y^2n_1^2} \gg \frac{T^2cm}{y^2n_1^2}$$

and $q'(y) \ll y^{-\frac{7}{3}}T^\varepsilon$, we have

$$v_1'(y)q'(y)^{-1} \gg y^{\frac{4}{3}}\frac{T^{2-\varepsilon}cm}{\sqrt{N}} \gg T^{\frac{5}{3}},$$

by partial integration many times, one shows that the contribution to (4.24) from the first integral in (4.26) is negligible.

Now we turn to $v_2(y)$ defined by (4.28). Since

$$v_2'(y) = -x^{\frac{1}{3}}y^{-\frac{2}{3}} + \frac{T^2cm}{4\pi^2y^2n_1^2},$$

if

$$(4.30) \quad x \geq \frac{T^6c^3m^3n_1^2}{10\pi^6N^2} \quad \text{or} \quad x \leq \frac{T^6c^3m^3n_1^2}{1000\pi^6N^2},$$

one has

$$|v_2'(y)| \gg \frac{T^2cm}{y^2n_1^2}.$$

As the arguments above, one shows that under the condition (4.30), the contribution to (4.24) from the second integral in (4.26) is negligible. For the remaining case

$$(4.31) \quad \frac{T^6c^3m^3n_1^2}{1000\pi^6N^2} \leq x \leq \frac{T^6c^3m^3n_1^2}{10\pi^6N^2}, \quad \text{i.e.,} \quad \frac{L_2}{1000} \leq l_2 \leq \frac{L_2}{10}$$

with

$$L_2 = \frac{T^6c^3m^3n_1^3c'^3}{\pi^6N^2l_1^2},$$

we have

$$|v_2''(y)| \gg \frac{T^2cm}{y^3n_1^2} \gg T^2cmN^{-\frac{3}{2}}n_1^4.$$

Therefore, by the second derivative test ([Hu], p. 88), one derives that

$$(4.32) \quad \begin{aligned} B_0(x) &\ll x^{\frac{2}{3}} \left(T^2cmN^{-\frac{3}{2}}n_1^4\right)^{-\frac{1}{2}} \left(\frac{\sqrt{N}}{n_1^2}\right)^{-\frac{4}{3}} T^\varepsilon \\ &\ll T^{3+\varepsilon}c^{\frac{3}{2}}N^{-\frac{5}{4}}n_1^2m^{\frac{3}{2}}. \end{aligned}$$

Combining (4.24), (4.25), (4.32) and invoking the trivial bound for the Kloosterman sum one concludes that

$$\begin{aligned} \tilde{\mathcal{R}}_3^+ &\ll MT \sum_{m \geq 1} m^{-1} \sum_{c \leq C_2/m} c^{-1} \sum_{n_1 | cm} n_1^{-1} \\ &\quad \times \sum_{u \pmod{mc n_1^{-1}}} (1 + un_1, c) c' \sum_{l_1 | c' n_1} \sum_{\frac{L_2}{100} \leq l_2 \leq \frac{L_2}{10}} \frac{|A(l_1, l_2)|}{l_1 l_2} \\ &\quad \times n_1 c' l_1^{-1} T^{3+\varepsilon} c^{\frac{3}{2}} N^{-\frac{5}{4}} n_1^2 m^{\frac{3}{2}} + O(MT^{1+\varepsilon}) \\ &\ll NT^{-\frac{1}{2}} M^{-\frac{7}{2}} + O(MT^{1+\varepsilon}) \ll MT^{1+\varepsilon} \end{aligned}$$

since $M \geq T^{\frac{3}{8}}$. This finishes the estimation of \mathcal{R}^+ .

5 The terms related to the K -Bessel function

This section is devoted to the estimation of \mathcal{R}^- which is defined by (2.23). We split \mathcal{R}^- into two parts \mathcal{R}_1^- and \mathcal{R}_2^- with

$$(5.1) \quad \mathcal{R}_1^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \geq C/m} c^{-1} S(n, 1; c) H_{m, n}^- \left(\frac{4\pi\sqrt{n}}{c}\right)$$

and

$$(5.2) \quad \mathcal{R}_2^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \leq C/m} c^{-1} S(n, 1; c) H_{m, n}^- \left(\frac{4\pi\sqrt{n}}{c}\right),$$

here

$$(5.3) \quad C = \sqrt{N} + T.$$

First we will estimate (5.1). By (2.24) and the following formula ([Wa], p. 78)

$$K_\nu(z) = \frac{1}{2} \pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu \pi}$$

where $I_\nu(z)$ is the I -Bessel function, we have

$$\begin{aligned} H_{m, n}^-(x) &= 2 \int_{-\infty}^{\infty} \frac{I_{-2it}(x) - I_{2it}(x)}{\sin 2it\pi} \sinh(\pi t) k(t) V(m^2 n, t) t dt \\ &= -4 \int_{-\infty}^{\infty} \frac{I_{2it}(x)}{\sin 2it\pi} \sinh(\pi t) k(t) V(m^2 n, t) t dt. \end{aligned}$$

Moving the line of integration to $\Im t = -\sigma = -100$, $H_{m, n}^-(x)$ becomes

$$\begin{aligned} (5.4) \quad &-4 \int_{-\infty}^{\infty} [\sin \pi(2\sigma + 2iy)]^{-1} I_{2\sigma+2iy}(x) \sinh \pi(-\sigma i + y) \\ &\quad \times k(-\sigma i + y) V(m^2 n, -\sigma i + y) (-\sigma i + y) dy. \end{aligned}$$

By the following formula ([GR], 8.431 3)

$$I_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{x \cos \theta} \sin^{2\nu} \theta d\theta$$

for $\Re \nu > -\frac{1}{2}$, one derives that

$$(5.5) \quad I_{2\sigma+2iy}(x) \ll_\sigma x^{2\sigma} |y|^{-2\sigma} e^{\pi y} e^x.$$

Combining (5.4), (5.5) and (4.7), we have

$$(5.6) \quad H_{m,n}^-(x) \ll x^{2\sigma} e^x (m^2 n)^{-\sigma} T^{\sigma+1+\varepsilon} M.$$

By (2.7), (5.5) and the trivial bound for the Kloosterman sum, one obtains that

$$\begin{aligned} \mathcal{R}_1^- &\ll \sum_{m \geq 1} \sum_{n \geq 1} \frac{|A(n, m)|}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \geq C/m} \left(\frac{\sqrt{n}}{c}\right)^{2\sigma} T^{\sigma+1+\varepsilon} (m^2 n)^{-\sigma} M \\ &\ll N^{\frac{1}{2}} T^{2-\sigma+\varepsilon} M \ll 1. \end{aligned}$$

It remains to estimate \mathcal{R}_2^- . By the following integral representation of the K -Bessel function (see [GR], 8.432 4)

$$K_{2it}(x) = \frac{1}{2} \cosh^{-1} t \pi \int_{-\infty}^{\infty} \cos(x \sinh \zeta) e\left(-\frac{t\zeta}{\pi}\right) d\zeta$$

and partial integration in ζ once, we have

$$\begin{aligned} H_{m,n}^-(x) &= \frac{4}{\pi} \int_0^\infty \int_{|\zeta| \leq T^\varepsilon} \tanh \pi t e^{-\frac{(t-T)^2}{M^2}} V(m^2 n, t) t \cos(x \sinh \zeta) \\ &\quad \times e\left(-\frac{t\zeta}{\pi}\right) d\zeta dt + O(T^{-A}) \end{aligned}$$

where A is arbitrarily large. By making change of a variable $\frac{t-T}{M} \rightarrow t$,

$$\begin{aligned} H_{m,n}^-(x) &= \frac{4M}{\pi} \int_{-\frac{T}{M}}^\infty \int_{|\zeta| \leq T^\varepsilon} \tanh \pi(tM + T) e^{-t^2} V(m^2 n, tM + T) \\ &\quad \times (tM + T) \cos(x \sinh \zeta) e\left(-\frac{tM\zeta}{\pi} - \frac{T\zeta}{\pi}\right) dt d\zeta + O(T^{-A}). \end{aligned}$$

Following the derivation of Proposition 4.1, by extending the t integral to $(-\infty, \infty)$ with a negligible error term, we have

$$H_{m,n}^-(x) = H_{m,n}^{-,1}(x) + H_{m,n}^{-,2}(x) + O(T^{-A}),$$

where

$$H_{m,n}^{-,1}(x) = \frac{4MT}{\pi} \int_{t=-\infty}^{\infty} \int_{|\zeta| \leq T^\varepsilon} e^{-t^2} V(m^2 n, tM + T) \cos(x \sinh \zeta) \\ \times e \left(-\frac{(tM + T)\zeta}{\pi} \right) dt d\zeta$$

and

$$H_{m,n}^{-,2}(x) = \frac{4M^2}{\pi} \int_{t=-\infty}^{\infty} \int_{|\zeta| \leq T^\varepsilon} te^{-t^2} V(m^2 n, tM + T) \cos(x \sinh \zeta) \\ \times e \left(-\frac{(tM + T)\zeta}{\pi} \right) dt d\zeta.$$

In the following we only treat $H_{m,n}^{-,1}(x)$. $H_{m,n}^{-,2}(x)$ is a lower order term which can be handled in a similar way. It is clear that

$$H_{m,n}^{-,1}(x) = \frac{4MT}{\pi} \int_{|\zeta| \leq T^\varepsilon} \hat{k}^* \left(\frac{M\zeta}{\pi} \right) \cos(x \sinh \zeta) e \left(-\frac{T\zeta}{\pi} \right) d\zeta$$

which is equal to

$$(5.7) \quad 4T \int_{|\zeta| \leq \pi^{-1}MT^\varepsilon} \hat{k}^*(\zeta) \cos \left(x \sinh \frac{\zeta\pi}{M} \right) e \left(-\frac{T\zeta}{M} \right) d\zeta$$

by making a change of variable $\frac{M\zeta}{\pi} \rightarrow \zeta$. Since $\hat{k}^*(\zeta)$ is a Schwartz class function, one can extend the above integral to $(-\infty, \infty)$ with a negligible error term. Now let

$$Y_{m,n}(x) := T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) \cos \left(x \sinh \frac{\zeta\pi}{M} \right) e \left(-\frac{T\zeta}{M} \right) d\zeta$$

and

$$(5.8) \quad Y_{m,n}^*(x) := T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) e \left(-\frac{T\zeta}{M} + \frac{x}{2\pi} \sinh \frac{\zeta\pi}{M} \right) d\zeta,$$

then

$$(5.9) \quad Y_{m,n}(x) = \frac{Y_{m,n}^*(x) + Y_{m,n}^*(-x)}{2}$$

and

$$H_{m,n}^{-,1}(x) = 4Y_{m,n}(x) + O(T^{-A})$$

with A arbitrarily large. Let

$$\Omega(\zeta) = \frac{x \sinh \frac{\zeta\pi}{M}}{2\pi} - \frac{T\zeta}{M},$$

then

$$\Omega'(\zeta) = \frac{x \cosh \frac{\zeta\pi}{M}}{2M} - \frac{T}{M}.$$

Then if

$$|x| \leq \frac{1}{100}T \quad \text{or} \quad |x| \geq 100T$$

then

$$\Omega'(\zeta) \gg \frac{T}{M} \gg T^\varepsilon,$$

hence by partial integrations,

$$Y_{m,n}^*(x) \ll T^{-A}$$

with $A > 0$ arbitrarily large. We are left with the case when

$$\frac{1}{100}T \leq x \leq 100T,$$

then

$$\frac{x}{M^3} \ll T^{-\frac{1}{8}}$$

(recall $M \geq T^{\frac{3}{8}}$). Now

$$\begin{aligned} Y_{m,n}^*(x) &= T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) e \left(-\frac{T\zeta}{M} + \frac{x\zeta}{2M} + \frac{\pi^2 x \zeta^3}{12M^3} + \frac{\pi^4 x \zeta^5}{240M^5} \right) d\zeta \\ &\quad + O \left(T \int_{-\infty}^{\infty} |\hat{k}^*(\zeta)| \frac{|\zeta|^7 |x|}{M^7} d\zeta \right). \end{aligned}$$

Expanding $e \left(\frac{\pi^2 x \zeta^3}{12M^3} + \frac{\pi^4 x \zeta^5}{240M^5} \right)$ into a Taylor series of order L_2 , we have

$$\begin{aligned} Y_{m,n}^*(x) &= T \int_{-\infty}^{\infty} \hat{k}^*(\zeta) e \left(-\frac{(2T-x)\zeta}{2M} \right) d\zeta \\ &\quad \times \sum_{l=0}^{L_2} \sum_{j=0}^l d_{j,l} \left(\frac{x\zeta^3}{M^3} \right)^j \left(\frac{x\zeta^5}{M^5} \right)^{l-j} d\zeta \\ &\quad + O \left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}} + \frac{T|x|}{M^7} \right) \end{aligned}$$

where $d_{j,l}$ are constants coming from the Taylor expansion and especially $d_{0,0} = 1$. Clearly

$$Y_{m,n}^*(x) = T \sum_{l=0}^{L_2} \sum_{j=0}^l d_{j,l} \frac{x^l}{M^{5l-2j}} k^{*(5l-2j)} \left(\frac{x-2T}{2M} \right) (2\pi i)^{-5l+2j} \\ + O \left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}} + \frac{T|x|}{M^7} \right).$$

We end up with the following proposition

Proposition 5.1. 1) For $|x| \geq 100T$ or $x \leq \frac{1}{100}T$,

$$Y_{m,n}^*(x) \ll T^{-A}$$

where $A > 0$ is arbitrarily large and the implied constant depends only on A .
2) For $\frac{1}{100}T \leq |x| \leq 100T$, $T^{\frac{3}{8}+\varepsilon} \leq M \leq T^{\frac{1}{2}}$ and $L_2 \geq 1$,

$$Y_{m,n}^*(x) = T \sum_{l=0}^{L_2} \sum_{j=0}^l b_{j,l} \frac{x^l}{M^{5l-2j}} k^{*(5l-2j)} \left(\frac{x-2T}{2M} \right) \\ + O \left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}} + \frac{T|x|}{M^7} \right),$$

where $b_{j,l}$ are constants depending only on j and l , especially $b_{0,0} = 1$.

The contribution to \mathcal{R}_2^- from the error term $O\left(\frac{T|x|}{M^7}\right)$ in the above proposition is $O(T^{1+\varepsilon}M)$ by (2.7) and the trivial bound for the Kloosterman sum. We always take L_2 sufficiently large such that the first error term in Proposition 5.1 2) is negligible. From now on we only take the leading term $l = 0$ since all the other lower order terms can be handled similarly. Let

(5.10)

$$\tilde{\mathcal{R}}_2^- := T \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n,m)}{(m^2n)^{\frac{1}{2}}} g\left(\frac{m^2n}{N}\right) \sum_{\frac{\sqrt{N}}{100Tm} \leq c \leq \frac{100\sqrt{N}}{Tm}} c^{-1} S(n,1;c) k^* \left(\frac{\frac{4\pi\sqrt{n}}{c} - 2T}{2M} \right).$$

If we sum over n trivially and apply Weil's bound for the Kloosterman sum, one derives that

$$\tilde{\mathcal{R}}_2^- \ll T^{\frac{1}{2}} N^{\frac{3}{4}+\varepsilon} \ll T^{\frac{11}{4}+\varepsilon}.$$

To save $T^{\frac{7}{4}}M^{-1}$, we have to sum over n nontrivially using the Voronoi formula for $GL(3)$. Expanding the Kloosterman sum in (5.10), by Proposition 2.1, we

have

$$\begin{aligned}
(5.11) \quad & \sum_{n \geq 1} A(n, m) e\left(\frac{n\bar{a}}{c}\right) r(n) \\
&= \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_1 n_2} S(ma, n_2; mc n_1^{-1}) R_{0,1}^0\left(\frac{n_2 n_1^2}{c^3 m}\right) \\
&+ \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_1 n_2} S(ma, -n_2; mc n_1^{-1}) R_{0,1}^1\left(\frac{n_2 n_1^2}{c^3 m}\right)
\end{aligned}$$

where

$$r(y) = g\left(\frac{m^2 y}{N}\right) k^*\left(\frac{\frac{4\pi\sqrt{y}}{c} - 2T}{2M}\right) y^{-\frac{1}{2}}$$

and $R_{0,1}^0(x)$ and $R_{0,1}^1(x)$ are defined below (2.5). As before, in the following, we only consider $R_0(x)$ since $x^{-1}R_1(x)$ has similar asymptotic behavior as of $R_0(x)$. Since

$$\frac{n_2 n_1^2}{c^3 m} \frac{N}{m^2} \gg \frac{T^3}{N^{\frac{1}{2}}} \gg T^{\frac{3}{2}-\varepsilon},$$

by Lemma 2.1 for $x = \frac{n_2 n_1^2}{c^3 m}$,

$$R_0(x) = 2\pi^4 x i \int_0^\infty r(y) \frac{d_1 \sin(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}})}{(\pi^3 x y)^{\frac{1}{3}}} dy + \text{lower order terms.}$$

If $n_2 \gg \frac{N^{\frac{1}{2}} T^\varepsilon}{M^3 n_1^2}$, then

$$x^{\frac{1}{3}} y^{-\frac{2}{3}} [r'(y)]^{-1} \gg T^\varepsilon.$$

By partial integration many times, one shows that the contribution to $\tilde{\mathcal{R}}_2^-$ from such terms is negligible. Next we assume

$$n_2 \ll \frac{N^{\frac{1}{2}} T^\varepsilon}{M^3 n_1^2}.$$

Since $k^*(y) \ll (1 + |y|)^{-A}$ for any $A > 0$, $r(y)$ is negligible unless

$$\left| \frac{\frac{2\pi\sqrt{y}}{c} - T}{M} \right| \leq T^\varepsilon$$

which implies that

$$\frac{1}{4\pi^2} (Tc - T^\varepsilon Mc)^2 \leq y \leq \frac{1}{4\pi^2} (Tc + T^\varepsilon Mc)^2,$$

then

$$(5.12) \quad R_0(x) \ll x^{\frac{2}{3}} \left(\frac{N}{m^2}\right)^{-\frac{5}{6}} T^{1+\varepsilon} M c^2.$$

Combining (5.10), (5.11), (4.22) and (5.12), we have

$$\begin{aligned} \tilde{\mathcal{R}}_2^- &\ll T \sum_{m \leq \sqrt{N}} \frac{1}{m} \sum_{\frac{\sqrt{N}}{100Tm} \leq c \leq \frac{100\sqrt{N}}{Tm}} \sum_{n_1 | cm} \sum_{n_2 \leq \frac{N^{\frac{1}{2}} T^\varepsilon}{M^3 n_1^2}} \\ &\quad \times \frac{|A(n_1, n_2)|}{n_1 n_2} m c^{1+\varepsilon} \left(\frac{n_2 n_1^2}{c^3 m} \right)^{\frac{2}{3}} \left(\frac{N}{m^2} \right)^{-\frac{5}{6}} T^{1+\varepsilon} M c^2 \\ &\ll N^{\frac{1}{2}} M^{-1} T^\varepsilon \ll T^{1+\varepsilon} M \end{aligned}$$

since $M \geq T^{\frac{3}{8}}$.

This finishes the estimation of \mathcal{R}^- and hence the proof of the main theorem.

Appendix

In this appendix, we consider the subconvexity problem of $L(\frac{1}{2}, f \times h)$ where f is a self dual Hecke-Maass form for $SL(3, \mathbb{Z})$ and h runs through holomorphic Hecke cusp forms of weight $k \geq 2$ and congruent to 0(mod 4) for $SL(2, \mathbb{Z})$. This analogous problem was suggested by Peter Sarnak and we would like to thank him here.

Let $\mathcal{B}_k(SL(2, \mathbb{Z}))$ denote an orthogonal basis of holomorphic Hecke cusp forms of weight $k \equiv 0 \pmod{4}$ for $SL(2, \mathbb{Z})$, each h in $\mathcal{B}_k(SL(2, \mathbb{Z}))$ is normalized to have the first Fourier coefficient $a_h(1)$ equal to 1. Set

$$\lambda_h(n) = \frac{a_h(n)}{n^{\frac{k-1}{2}}}.$$

By Deligne [De],

$$|\lambda_h(n)| \leq \tau(n).$$

For f a self dual Hecke-Maass form of type (ν, ν) for $SL(3, \mathbb{Z})$ with the Fourier-Whittaker expansion (2.4) and $h \in \mathcal{B}_k(SL(2, \mathbb{Z}))$, we define the Rankin-Selberg L -function

$$L(s, f \times h) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_h(n) A(n, m)}{(m^2 n)^s}.$$

It is entire and satisfies the functional equation

$$(A.1) \quad \Lambda(s, f \times h) = \Lambda(1-s, f \times h)$$

where

$$\begin{aligned} \Lambda(s, f \times h) &= \pi^{-3s} \Gamma\left(\frac{s + \frac{k-1}{2} - \alpha}{2}\right) \Gamma\left(\frac{s + \frac{k-1}{2} - \beta}{2}\right) \Gamma\left(\frac{s + \frac{k-1}{2} - \gamma}{2}\right) \\ &\quad \times \Gamma\left(\frac{s + \frac{k+1}{2} - \alpha}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2} - \beta}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2} - \gamma}{2}\right) L(s, f \times h) \end{aligned}$$

and

$$(A.2) \quad \alpha = -3\nu + 1, \quad \beta = 0, \quad \gamma = 3\nu - 1.$$

The above functional equation can be obtained by examining the template arising from the case of the minimal parabolic Eisenstein series for $GL(3)$ twisted by a cusp form in $\mathcal{B}_k(SL(2, \mathbb{Z}))$ (see [Gol], p. 315). Note the sign of the above functional equation is $+1$ because we restrict k to be congruent to $0 \pmod{4}$ (see [IK] p. 131 and [Iw1] p. 121). This is important because we need the uniformity of the sign of the functional equations of $L(\frac{1}{2}, f \times h)$ when applying the Petersson formula. The main theorem in this appendix is

Theorem A.1. *Let f be a fixed self dual Hecke-Maass form for $SL(3, \mathbb{Z})$, then for $\varepsilon > 0$, K large and $K^{\frac{3}{8}+\varepsilon} \leq M \leq K^{\frac{1}{2}}$, we have*

$$\sum_{2 \leq k \equiv 0 \pmod{4}} e^{-\frac{(k-K)^2}{M^2}} \sum_{h \in \mathcal{B}_k(SL(2, \mathbb{Z}))} L\left(\frac{1}{2}, f \times h\right) \ll_{\varepsilon, f} K^{1+\varepsilon} M.$$

As we explained in the introduction, Lapid's theorem applies which means that $L(\frac{1}{2}, f \times h) \geq 0$. Due to this important property, we have

Corollary A.1. *Under the same assumptions as in the above theorem,*

$$L\left(\frac{1}{2}, f \times h\right) \ll_{\varepsilon, f} k^{\frac{11}{8}+\varepsilon}.$$

The corresponding convexity bound for $L(\frac{1}{2}, f \times h)$ is $k^{\frac{3}{2}+\varepsilon}$ with $\varepsilon > 0$, so the above bound breaks the convexity bound. The rest of the paper is devoted to the proof of Theorem A.1. As in Lemma 2.2, we have the following approximate functional equation for $L(s, f \times h)$:

$$(A.3) \quad L\left(\frac{1}{2}, f \times h\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(n, m)}{(m^2 n)^{\frac{1}{2}}} U(m^2 n, k)$$

where

$$U(y, k) = \frac{1}{2\pi i} \int_{(1000)} y^{-u} F(u) \frac{\gamma(\frac{1}{2} + u, k)}{\gamma(\frac{1}{2}, k)} \frac{du}{u}$$

and

$$\begin{aligned} \gamma(s, k) = \pi^{-3s} & \Gamma\left(\frac{s + \frac{k-1}{2} - \alpha}{2}\right) \Gamma\left(\frac{s + \frac{k-1}{2} - \beta}{2}\right) \Gamma\left(\frac{s + \frac{k-1}{2} - \gamma}{2}\right) \\ & \times \Gamma\left(\frac{s + \frac{k+1}{2} - \alpha}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2} - \beta}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2} - \gamma}{2}\right). \end{aligned}$$

We introduce the spectrally normalized first moment of the central values of L -functions

$$\mathcal{A} := \sum_{2 \leq k \equiv 0 \pmod{4}} e^{-\frac{(k-K)^2}{M^2}} \sum_{h \in \mathcal{B}_k(SL(2, \mathbb{Z}))} \frac{KL(\frac{1}{2}, f \times h)}{(k-1)L(1, \text{sym}^2 h)}.$$

The weights $L^{-1}(1, \text{sym}^2 h)$ are needed in the Petersson formula and they are harmless since it is known ([Iw], [HL]) that

$$k^{-\varepsilon} \ll L(1, \text{sym}^2 h) \ll k^\varepsilon$$

for any $\varepsilon > 0$. Applying (A.1) to \mathcal{A} , it is enough to show

$$(A.4) \quad \sum_{2 \leq k \equiv 0 \pmod{4}} e^{-\frac{(k-K)^2}{M^2}} \frac{K}{k-1} \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} U(m^2 n, k) \\ \times g\left(\frac{m^2 n}{N}\right) \mathcal{F}_k \ll K^{1+\varepsilon} M,$$

here g is a fixed smooth function of compact support on $[1, 2]$, $1 \leq N \ll_\varepsilon K^{3+\varepsilon}$ and

$$\mathcal{F}_k = \sum_{h \in \mathcal{B}_k(SL(2, \mathbb{Z}))} \frac{\lambda_h(n)}{L(1, \text{sym}^2 h)}.$$

By Petersson's formula (see [ILS], p. 111, for example),

$$(A.5) \quad \mathcal{F}_k = \frac{k-1}{2\pi^2} \left[\delta(n, 1) + 2\pi \sum_{c \geq 1} c^{-1} S(n, 1; c) J_{k-1} \left(\frac{4\pi\sqrt{n}}{c} \right) \right].$$

We then write the left side of (A.4) as

$$\mathcal{D}_w + \mathcal{N}\mathcal{D}_w,$$

where

$$(A.6) \quad \mathcal{D}_w = \sum_{2 \leq k \equiv 0 \pmod{4}} \frac{K}{2\pi^2} e^{-\frac{(k-K)^2}{M^2}} \sum_{m \geq 1} \frac{A(1, m)}{m} U(m^2, k) g\left(\frac{m^2}{N}\right)$$

and

$$(A.7) \quad \mathcal{N}\mathcal{D}_w = \sum_{2 \leq k \equiv 0 \pmod{4}} \frac{K}{\pi} e^{-\frac{(k-K)^2}{M^2}} \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(n, m)}{(m^2 n)^{\frac{1}{2}}} U(m^2 n, k) \\ \times g\left(\frac{m^2 n}{N}\right) \sum_{c \geq 1} c^{-1} S(n, 1; c) J_{k-1} \left(\frac{4\pi\sqrt{n}}{c} \right).$$

From (3.2),

$$\mathcal{D}_w \ll K^{1+\varepsilon} M,$$

which is consistent with the desired bound in (A.2).

To estimate $\mathcal{N}\mathcal{D}_w$, we begin by executing the k -sum by Poisson summation as in [Iw1] (p. 86) and [Sa] (p. 430). Applying the following integral representation [GR] of the J -Bessel function

$$J_l(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e(lt) e^{-ix \sin 2\pi t} dt$$

and the Poisson summation in k yields

$$(A.8) \quad K \sum_{2 \leq k \equiv 0 \pmod{4}} u(k-1) J_{k-1}(x) = -\frac{1}{2} V_1(x) + \frac{i}{2} V_2(x)$$

where $u(x) = e^{-\frac{(x+1-K)^2}{M^2}} U(m^2 n, x+1)$,

$$(A.9) \quad V_1(x) = K \int_{-\infty}^{\infty} \hat{u}(t) \sin(x \cos 2\pi t) dt,$$

and

$$(A.10) \quad V_2(x) = K \int_{-\infty}^{\infty} \hat{u}(t) \sin(x \sin 2\pi t) dt,$$

with $\hat{u}(t)$ be the Fourier transform of $u(x)$ as defined in (4.12). Since

$$\hat{u}(t) = M e(-(K-1)t) \hat{u}_0(Mt)$$

with

$$(A.11) \quad u_0(x) = e^{-x^2} U(m^2 n, xM + K),$$

we have

$$(A.12) \quad V_1(x) = K \int_{-\infty}^{\infty} \hat{u}_0(t) e\left(\frac{-(K-1)t}{M}\right) \sin\left(x \cos \frac{2\pi t}{M}\right) dt,$$

and

$$(A.13) \quad V_2(x) = K \int_{-\infty}^{\infty} \hat{u}_0(t) e\left(\frac{-(K-1)t}{M}\right) \sin\left(x \sin \frac{2\pi t}{M}\right) dt.$$

We will first estimate the contribution to (A.7) from $V_1(x)$. Set

$$V_1^*(x) = K \int_{-\infty}^{\infty} \hat{u}_0(t) e\left(\frac{-(K-1)t}{M} - \frac{x}{2\pi} \cos \frac{2\pi t}{M}\right) dt,$$

then

$$V_1(x) = \frac{V_1^*(-x) - V_1^*(x)}{2i}.$$

One can see that $V_1^*(x)$ and $W_{m,n}^*(x)$ (see (4.14)) have similar integral representation. Following the derivation of Proposition 4.1, it is straightforward to derive the following:

Proposition A.2. 1) For $|x| \leq K^{1-\varepsilon}M$ with $\varepsilon > 0$,

$$V_1^*(x) \ll K^{-A}$$

where $A > 0$ is arbitrarily large and the implied constant depends on ε and A .

2) For $K^{1-\varepsilon}M \leq |x| \leq M^4$, $K^{\frac{3}{8}+\varepsilon} \leq M \leq K^{\frac{1}{2}}$ and $L_2, L_1 \geq 1$,

$$\begin{aligned} V_1^*(x) = & \frac{KM}{\sqrt{|x|}} e \left(\frac{-x}{2\pi} + \frac{(K-1)^2}{4\pi x} \right) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{\frac{l_1}{4} \leq l_2 \leq L_2} c_{l,l_1,l_2} \frac{M^{2l-l_1} (K-1)^{4l_2-l_1}}{x^{l+3l_2-l_1}} \\ & \times \left[\hat{u}_0^{(2l-l_1)} \left(\frac{(K-1)M}{2\pi x} \right) + \frac{4\pi^6 i x}{45M^6} (t^6 \hat{u}_0(t))^{(2l-l_1)} \left(\frac{(K-1)M}{2\pi x} \right) \right] \\ & + O \left(\frac{KM}{\sqrt{|x|}} \left(\frac{T^4}{|x|^3} \right)^{L_2+1} + K \left(\frac{M}{\sqrt{|x|}} \right)^{2L_1+3} + \frac{K|x|}{M^8} \right) \end{aligned}$$

where c_{l,l_1,l_2} are constants depending only on l, l_1 and l_2 .

Now we consider the contribution to (A.5) from $V_2(x)$ given by (A.13). Set

$$V_2^*(x) = K \int_{-\infty}^{\infty} \hat{u}_0(t) e \left(\frac{-(K-1)t}{M} + \frac{x}{2\pi} \sin \frac{2\pi t}{M} \right) dt,$$

then

$$V_2(x) = \frac{V_2^*(x) - V_2^*(-x)}{2i}.$$

One can see that $V_2^*(x)$ and $Y_{m,n}^*(x)$ (see (5.8)) have similar integral representation, so they have similar asymptotic behavior (see Proposition (5.1)):

Proposition A.3. 1) For $|x| \geq 100K$ or $|x| \leq \frac{1}{100}K$,

$$V_2^*(x) \ll K^{-A}$$

where $A > 0$ is arbitrarily large and the implied constant depends only on A .

2) For $\frac{1}{100}K \leq |x| \leq 100K$, $K^{\frac{3}{8}+\varepsilon} \leq M \leq K^{\frac{1}{2}}$ and $L_2 \geq 1$,

$$\begin{aligned} V_2^*(x) = & K \sum_{l=0}^{L_2} \sum_{j=0}^l a_{j,l} \frac{x^l}{M^{5l-2j}} u_0^{(5l-2j)} \left(\frac{x-K+1}{M} \right) \\ & + O \left(\frac{K|x|^{L_2+1}}{M^{3L_2+3}} + \frac{K|x|}{M^7} \right), \end{aligned}$$

where $a_{j,l}$ are constants depending only on j and l .

Replacing T by $(K-1)/2$ and k^* by u_0 in sections 4 and 5, one can see that Theorem A.1 follows directly from Propositions A.2 and A.3.

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